Belief-Consistent-Pareto Dominance

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September 2017

Abstract

Classic Pareto criterion claims that every voluntary trades even on the ground of heterogenous beliefs should be encouraged. I argue that a trade without hope for Pareto improvement remains controversial. I introduce and characterize a notion of *Belief-Consistent-Pareto* dominance to formalize this argument, which requires, aside from unanimity of preferences, all rankings in a trade should be supported by some common beliefs, which have to coincide with agents' beliefs over the events on which all agents agree.

1 Introduction

For long, free Pareto-improving trades have received almost faithful respect from economists. In a democratic society, how should traders be denied their mutual desire to trade. In recent years, this principle has gained much less support in the face of uncertainty, in particular, when the involved traders possess different beliefs on which the trade is hinged. Examples laid out by Weyl [2007] and Kreps [2012] demonstrated the impairment of free trade when traders have different beliefs.[1] Guided by similar observations, Gilboa, Samuelson, and Schmeidler [2014], Brunnermeier, Simsek, and Xiong [2014], Posner and Weyl [2013] and Blume, Cogley, Easley, Sargent, and Tsyrennikov [2014], among many others, claim that society may be better off with restrictions that prevent such malicious trades from taking place.

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[1] In advance, preference aggregation with heterogeneous beliefs have similar flavor. Gilboa, Samet, and Schmeidler [2004] offered an example in which two parties do not receive mutual gains from a duel but are expecting to benefit at the expense of the other. Since both cannot be correct, Mongin [1997] argued that such unanimity is spurious.

This paper proposes a *belief-consistent-Pareto* criterion, which can somehow justify benevolent paternalism whenever traders holding heterogeneous beliefs give rise to controversial trades. To illustrate the basic idea, consider the following adaptation of an example in Kreps [2012].[2] There are two agents, Alice and Bob, who argue about the contents of a pillow. Suppose this pillow only has three possible types of contents, classic feather, polyester and goose down. Both Alice and Bob know goose down pillow is a limited edition, and, therefore, share the belief that with probability 0.1 that the pillow had a goose down filling. However, Alice and Bob maintain different beliefs about other possibilities. In particular, Alice believes pillow contains classic feather with probability 0.1 and polyester with probability 0.8. Bob instead believes that pillow contains classic feather with probability 0.8 and polyester with probability 0.1. I assume that whether the pillow is limited edition or not is revealed once the package is open. However, if the pillow is not limited edition, to know whether the content is classic feather or polyester, the pillow has to be cut openly. Alice and Bob decide to construct a bet as follows: if the pillow is a limited edition, they will resale it for \$150 and share it equally, but if it had classic feather content, Alice would pay Bob \$10, similarly if it had polyester content, Bob would pay Alice \$10. They agree that they will share the price \$90 of pillow equally if it is a limited edition, otherwise loser would pay it alone.

There might be some behavioral explanation about why Alice and Bob desire to take the bet. In this paper, I only invoke Bayesian rationality throughout the analysis. Suppose both are risk-neutral. Alice and Bob would agree to bet, each believing they are correct in beliefs and taking advantage of the other. Should this bet be regarded as a Pareto improvement and, therefore, be encouraged? The standard Pareto improvement view among economists require common beliefs among all agents. In the setting of heterogeneous beliefs, the meaning of Pareto improvement turns out to be ambiguous and needs to be modified. For lack of common beliefs, Pareto improvement has to be evaluated under some *hypothetical* common beliefs—which cannot be proven wrong ex ante—such that if all agents hold such beliefs, the bet is still preferred by all agents. In other words, a Pareto improving bet under heterogenous beliefs is a bet that maintains a hope for Pareto improvement according to standard view. In my pillow example, any potentially reasonable beliefs should be consistent with the probability 0.1 that pillow is a limited edition. For any such beliefs, if both agents maintain these beliefs, at least one agent would be worse off under the bet. Therefore, allowing agents to bet in the example should not be viewed as a Pareto improvement. In this regard, a society should discourage agents from such bet, meaning a bet with the aim of profiting at the expense of the other, but without hope for Pareto improvement.

To generalize the key insight of this example, I reckon that a trade without hope for Pareto

[2] See page 193 of Kreps [2012] for the details of the story.

improvement on the ground of consistent common beliefs remains controversial. This paper proposes a notion of *Belief-Consistent-Pareto* dominance, a refinement of the standard notion of Pareto dominance. I offer the following criterion: an allocation f *Belief-Consistent-Pareto* dominates allocation g not only if all agents prefer f to g, but also if there exist some consistent common beliefs—which coincide with agents' beliefs over events that all agents agree on—such that if all agents hold these beliefs, all agents are better off under allocation f.

My criterion consists of two parts. The first is a standard Pareto dominance criterion and requires that each agent voluntarily trades allocation g for f. The second part requires that the trade has to be Pareto improving. In other words, there should exist a potentially reasonable belief such that if all agents hold such hypothetical beliefs, allocation f remains superior to g. It is worth to mention that a key presumption of hypothetical beliefs is that they should not be any arbitrary ones and have to be consistent. For consistent beliefs, I mean beliefs that coincide with agents' beliefs over the events on which all agents agree. Let me trot out the example again. Consider probabilities 0.2, 0.7, and 0.1 that, respectively, the content of pillow is classic feather, goose down and polyester. Should they hold these beliefs, Alice and Bob still prefer to bet. However, such hypothetical beliefs violate the evidence that with probability 0.1 the content is goose down. According to my criterion, any beliefs whose probability is not 0.1 on goose down content cannot be a candidate belief used to justify the bet. Since for any consistent common belief, at least one agent would be worse off under the bet, a society should disallow the bet.

So far I have argued that a voluntary trade with a hope for Pareto improvement should be respected. Now the more relevant question might be arguably whether only the consistent common beliefs should be applied to somehow justify paternalistic intervention. Before discussing the role of consistent common beliefs, it is useful to briefly review the underline interpretations of probabilistic beliefs in economic context. Probabilities that are pervasive in economics are categorized as either objective or subjective.[3] Objective probabilities are often used to measure some repetitive events or events generated by some random devices. In general, objective probabilities can be proved to be reasonably close to truth and are independent of any agents. In contrast, subjective probability measures the confidence that a particular agent has in the truth of a particular proposition. The two types of probabilities are obviously mutually exclusive. Therefore, the probability of a particular event we concern must be categorized as either the long run frequency or as the willingness to bet of some agent. In fact, most important economic contexts involve both types of events simultaneously. Thus, the events on which the agents' probabilities agree can be either

[3] Savage [1972] has identified three views of probability: objective, subjective and necessary. Since it is not clear how to estimate the necessary probability, its application in economics are very limited. I will not discuss it here for relevance and refer to Savage [1972] for details.

or both types of events. As for the unanimous events whose probabilities are objectively known, the principle of internal consistency as well as common intuition would imply the use of objective probabilities (see, for example, p.6 of Allais [1953].[4]). As for the unanimous events whose probabilities are subjective, no one can prove that such probability estimates are even close to the truth. Then, one may ask why I exhort a society to accept and actually use the unanimous subjective probabilities, in much the same way that it accepts the use of objective probabilities? The most basic reason I believe is that the fundamental rationale for paternalism does not apply to the unanimous subjective probabilities, since social intervention is typically defended on the grounds of conflicting beliefs. Far fewer would continue to pledge for intervention when agents have identical beliefs. Therefore, the fact that unanimous beliefs over some events substantially weakens the reasoning for belief intervention toward those events. Another important caveat that unanimous beliefs should be respected is from Savage. At one point he wrote that "the personalistic view incorporates all . . . criteria for reasonableness in judgment known to me, and . . . when any criteria that many have been overlooked are brought forward, they will be welcomed. . . " (Savage [1972], p. 67). Analogy to it, if the agents are rational in the sense of reasoning based on logic and probability, a society who is absent additional superior information should welcome such unanimous subjective probabilities.[5]

Several other economists have already proposed other criterion to justify paternalistic intervention for voluntary trading. Among those, the most related but distinct argument is proposed in Gilboa et al. [2014]. They introduced a notion of *No-Betting-Pareto* dominance, also a refinement of standard Pareto criterion. Under their notion, an allocation is regarded as Pareto dominance not only if all agents prefer this allocation, but also there exist some common beliefs—which can be selected without any constraint—such that all agents remain the ranking under such beliefs. The essential difference to my criterion is that they allow a society to pick an arbitrary belief to justify paternalism, regardless of whether it is consistent to unanimous agents' beliefs over some events. According to my analysis above, their arguments have their merits, but remain doubts in terms of belief selection for intervention. Two very similar arguments are laid out in Brunnermeier et al. [2014] and Gayer, Gilboa, Samuelson, and Schmeidler [2014]. According to their definition, an allocation is superior if it can be verified that no agents will be worse off if their expected utilities are evaluated by each agent's beliefs or any mixture of agents' beliefs. Both criterion are stronger than my criterion and, as a consequence, they would disallow much more trades. They suggest intervention with the aim of cautiousness and security, but may result in the risk of depressing

[4] An objection to such view appears in Morris [1995].

[5] See Aumann [1987] for further discussion about this argument. An objection to such view in the preference aggregation setting appears in Mongin and Pivato [2016].

subsequent economic activity.

Section 2 contains the main result. Some policy issues do arise and I discuss it in Section 3 as a conclusion. The Appendix contains the proof.

2 Main Result

There is a set of individuals $\mathcal{I} = \{1, \ldots, I\}$, a measurable state space (S, Σ), and a set of outcomes X. A generic outcome x specifies all the features related to all individuals. Typically, though not necessary for our purpose, we assume that $x = (x_1, \ldots, x_I) \in X \subseteq \mathbf{R}^I$, specifies a wealth level, x_i for each individual i. A set of allocations $\mathcal{F}$ consists of simple acts, which are measurable functions from states to outcomes.

Each individual i has a preference order $\succsim_i$ over $\mathcal{F}$. We assume that the individuals are expected utility maximizers. That is, each individual i has a utility function $u_i : X \to \mathbf{R}$ and a probability measure p_i on (S, Σ) such that for any pair of allocations f and g,

$$f \succsim_i g \Longleftrightarrow \int_S u_i(f(s))\mathrm{d}p_i \geq \int_S u_i(g(s))\mathrm{d}p_i.$$

A trade is a pair of allocation $(f, g) \in \mathcal{F} \times \mathcal{F}$, in which individuals exchange allocation g for f. It is not necessary that each individual involves in such trade. Individual $i \in \mathcal{I}$ is said to be a *trader* in (f, g) if there exists at least one state s at which the individual is not indifferent between two constant allocations, $f(s)$ and $g(s)$. Let $\mathcal{I}(f, g) \subseteq \mathcal{I}$ denote the set of traders in the trade (f, g). Let Δ denote all the probability measures on (S, Σ). The notion of Pareto dominance, denoted by $\succ$, and the notion of No-betting-Pareto dominance of Gilboa et al. [2014] (GSS), denoted by $\succ_{\text{NBP}}$, are defined as follows:

Definition 1. For a trade (f, g), we say that

1. $f \succ g$ if for all $i \in \mathcal{I}$, $f \succsim_i g$ and for all $j \in \mathcal{I}(f, g)$, $f \succ_j g$.[6]

2. $f \succ_{\text{NBP}} g$ if $f \succ g$ and there exists a $p_0 \in \Delta$ such that for all $i \in \mathcal{I}(f, g)$,

$$\int_S u_i(f(s))\mathrm{d}p_0 \geq \int_S u_i(g(s))\mathrm{d}p_0.$$

[6] Our notion of Pareto dominance, following directly from GSS, is more restrictive than standard Pareto dominance, which states that allocation f Pareto dominates g if $f \succsim_i g$ for all i, and $f \succ_j g$ for some j. In contrast, our notion requires that all traders have to be strictly prefer to f, which rules out the possibility to be indifferent for traders. We refer to p1416 of GSS for further elaboration.

Let $\Lambda = \{E \in \Sigma \big| \text{ for all } i, j \in \mathcal{I}, p_i(E) = p_j(E)\}$. Thus, an event E is *unanimous* if all individuals agree on its probability. The set of unanimous probability measures on (S, Σ) is defined by

$$\Delta(\Lambda) = \{p \in \Delta \big| \text{ for all } E \in \Lambda, p(E) = p_i(E) \text{ for all } i\}.$$

Our stronger notion of Pareto dominance is defined as follows:

Definition 2. For a trade (f, g), we say that allocation f *Belief-consistently* Pareto dominates g, denoted $f \succ_{\text{BCP}} g$, if:

(i) $f \succ g$;

(ii) there exists a probability measure $p^* \in \Delta(\Lambda)$ such that for all $i \in \mathcal{I}(f, g)$,

$$\int_S u_i(f(s))\mathrm{d}p^* \geq \int_S u_i(g(s))\mathrm{d}p^*.$$

Observe that same as the standard Pareto domination, our definition does not require that individuals coincide with the distributions of the allocations f and g according to their personal beliefs. Actually, the individual beliefs, which are eliminated from their preferences, might be quite different.

Condition (i) of the definition requires that all individuals weakly prefer allocation f to g and all traders strictly prefer f to g. Condition (ii) requires that one can find a unanimous probability measure, according to which all traders still prefer allocation f to g. That is, one can find a probability measure such that not only its probability over unanimous events is consistent with that of individuals, but also this conjectured belief preserves the traders' ranking, therefore, rationalize the trade.

Notice that we require the existence of unanimous probability measure to rationalize the trade (f, g). Our notion is thus more restrictive than No-Betting-Pareto domination, which only asks for the existence of probability measure, irrespective of unanimous or not. In contrast, out notion eliminates some hypothetical belief candidates, which we find unreasonable. We only allow the unanimous probability measures as the potential hypothetical beliefs to rationalize the trade. In particular, we are reluctant to assume that there exists a decent odds that individuals have fallacious beliefs over unanimous events. Again, our condition (ii) fails in our example above, where the trade can be 'rationalized' by a probability measure, but not a unanimous probability measure.

To generalize our example, we want to eliminate some technical complications by restricting the set of unanimous events. Consider a finite partition $\{E_1, \ldots, E_N\}$ of state space (S, Σ). We assume that Λ consists of all unions of the events E_n.[7] The following theorem characterizes the trades in terms of BCP.

Theorem 1. *Consider a trade (f, g) with $\mathcal{I}(f, g) \neq \emptyset$. Then the following are equivalent:*

(i) *There exists a probability measure $p^* \in \Delta(\Lambda)$ such that for all $i \in \mathcal{I}(f, g)$,*

$$\int_S u_i(f(s))\mathrm{d}p^* > \int_S u_i(g(s))\mathrm{d}p^*.$$

(ii) *For every distribution over the traders, $\lambda \in \Delta(\mathcal{I}(f, g))$, there exist states $s_n \in E_n$ for $n = 1, \ldots, N$, such that for $p \in \Delta(\Lambda)$,*

$$\sum_{i \in \mathcal{I}(f,g)} \lambda_i \cdot \Big(\sum_{n \leq N} p(E_n) u_i(f(s_n)) \Big) > \sum_{i \in \mathcal{I}(f,g)} \lambda_i \cdot \Big(\sum_{n \leq N} p(E_n) u_i(g(s_n)) \Big).$$

To interpret the result, consider the set of traders $\mathcal{I}(f, g)$. Presumably, for every partition event E_n, each trader can select a state s_n such that if s_n is the only non-null element in E_n, then the expected utility of f is higher that of g. The result states that f belief-consistently Pareto dominates g is equivalent to the convex combination of every such expected utility of traders, where the combination is defined by λ. Analogy to aggregation theory, λ can be regarded as the weights that a society assigns to each traders' expected utility.

3 Conclusion

This paper proposes a new notion of Pareto dominance based on heterogeneous individuals' beliefs. The idea behind my criterion is that to avoid the frequent intervention to deteriorate economic activity, any voluntary trades with a hope for Pareto improvement should be encouraged. In essence, only consistent beliefs can be used to justify the Pareto improvement. This idea can capture the nature of malicious trades that is a concern for a society. With regard to direct policy implications, my analysis highlights various difficulties in using my principle. Although I provide the conditions for when a society should intervene, the implementation comes with difficulties. For example, a society may have to know what are agents' utilities as well as their beliefs. Other justifications for intervention, for instance Brunnermeier et al. [2014], have the same difficulties.

[7] Although the collection of such unanimous events is in general a λ-system, our simplification is without loss of generality. One can assume a general λ-system. For every possible partition generated by this λ-system, one can apply Theorem 1. Therefore, it will increase the computational complexity without adding further insights.

Gilboa et al. [2014] do not require the information for agents' beliefs and regard it as a implementation advantage. Unfortunately, without information of beliefs, a society might suffer in mistaken beliefs and get into muddle. Although detecting beliefs and utilities is likely to be difficult, our analysis would still help tackle these issues in a rhetorical manner.

APPENDIX: PROOFS

Proof of Theorem. We know that a finite partition $\{E_1, \ldots, E_N\}$ generates the set of unanimous events Λ. We also know that probability p on Λ satisfies that $p(E) = p_i(E)$ for each i and E and Λ.

Pick a pair of allocations f and g. Since both allocations are simple and measurable, there is also a finite partition of S, $(A_j)_{j\le J}$, such that both f and g are constant over each A_j. For each $n \le N$ and $j \le J$, we define

$$B_{nj} = E_n \cup A_j.$$

Therefore, $(B_{nj})_{n\le N; j\le J}$ consists a coarser finite partition of S. Thus, we use $f(B_{nj})$ and $g(B_{nj})$ to denote the outcomes in X that f and g, respectively, assume over event B_{nj} for each $n \le N$ and $j \le J$.The definition of Belief-consistent Pareto dominance requires that there exists a probability measure $p^* \in \Delta(\Lambda)$ such that for each $i \in \mathcal{I}(f, g)$,

$$\int_S u_i(f)\mathrm{d}p^* > \int_S g_i(f)\mathrm{d}p^* \tag{1}$$

It is equivalent to the existence of a probability vector $p^* \in \Delta(N \times J)$ such that

$$\sum_{\substack{n\le N\\ j\le J}} p^*(n,j)u_i(f(B_{nj})) > \sum_{\substack{n\le N\\ j\le J}} p^*(n,j)u_i(g(B_{nj})) \text{ and } \sum_{j\le J} p^*(n,j) = p(E_n), \text{for all } n \le N. \tag{2}$$

Notice that if a probability measure $p^* \in \Delta(\Lambda)$ exists, it can reduce to a probability vector $p^*(n, j)$ satisfying Eq(2). Conversely, if a probability vector $p^*(n, j)$ exists, we can extend it to a unanimous probability measure on (S, Σ), which satisfies Eq(1).

To prove the existence of unanimous probability measure, we construct a two-person zero-sum game. For each $n \le N$, let $B^n = \{B_{n1}, \ldots, B_{nJ}\}$. The strategy for player I is

$$(b^1, \ldots, b^N) \in B = B^1 \times \cdots \times B^N.$$

The strategy for player II is $i \in \mathcal{I}(f, g)$. Given a strategy profile (b, i), the payoff for player I is defined as:

$$U(b,i)=\sum_{n=1}^{N}p(E_n)[u_i(f(b^n))-u_i(g(b^n))].$$

Player I may use mixed strategy and a simplex $\Delta(B)$ represents the set of mixed strategy for player I. Therefore, should player I chooses mixed strategy σ and player II chooses i, the payoff for player I is:

$$U(\sigma,i)=\sum_{(b^1,\ldots,b^N)\in B}\sigma(b^1,\ldots,b^N)\big[\sum_{n=1}^{N}p(E_n)(u_i(f(b^n))-u_i(g(b^n)))\big].$$

For any event B_{nj}, we denote

$$\sigma(B_{nj})=\sum_{\substack{b^n\neq B_{nj}\\(b^1,\ldots,b^N)\in B}}\sigma(b^1,\ldots,b^N).$$

Notice that $\sum_{j\leq J}\sigma(B_{nj})=1$. Therefore, the payoff for player I given strategy profile (σ,i) can be written as

$$U(\sigma,i)=\sum_{n\leq N}p(E_n)\sum_{j\leq J}\sigma(B_{nj})[u_i(f(B_{nj})-u_i(g(B_{nj})]$$

Then Eq(2) is equivalent to the existence of a mixed strategy of player I, $\sigma\in\Delta(B)$ such that, for every pure strategy of player II, $i\in\mathcal{I}(f,g)$,

$$U(\sigma,i)>0.$$

(To see the equivalence, simply define $p^*(n,j)=p(E_n)\cdot\sigma(B_{nj})$. Then $\sum_{j\leq J}p^*(n,j)=p(E_n)\cdot\sum_{j\leq J}\sigma(B_{nj})=p(E_n)$.)

Furthermore, Eq(2) is equivalent to the existence of $\sigma\in\Delta(B)$ such that, for every mixed strategy of play II, $\lambda\in\Delta(\mathcal{I}(f,g))$,

$$\sum_{i\in\mathcal{I}(f,g)}\lambda(i)\cdot U(\sigma,i)>0.$$

In other words,

$$\max_{\sigma\in\Delta(B)}\min_{\lambda\in\Delta(\mathcal{I}(f,g))}\sum_{i}\lambda(i)\cdot U(\sigma,i)>0.$$

According to the minmax theorem for two-person zero-sum games, we have

$$\min_{\lambda\in\Delta(\mathcal{I}(f,g))}\max_{\sigma\in\Delta(B)}\sum_i \lambda(i)\cdot U(\sigma,i)>0.$$

That is to say, Eq(2) holds if and only if for any λ, there exists $\sigma\in\Delta(B)$ such that

$$\sum_{i\in\mathcal{I}(f,g)}\lambda(i)\sum_{(b^1,\ldots,b^N)\in B}\sigma(b^1,\ldots,b^N)\Big[\sum_{n=1}^{N}p(E_n)(u_i(f(b^n))-u_i(g(b^n)))\Big]>0.$$

Notice that for each λ, such a $\sigma\in\Delta(B)$ exists if and only if there exists a unit vector, namely, there exists $(b^1,\ldots,b^N)$ such that

$$\sum_{i\in\mathcal{I}(f,g)}\lambda(i)\sum_{n=1}^{N}p(E_n)(u_i(f(b^n))-u_i(g(b^n)))>0.$$

This is the same case that there exist $s_n\in E_n$ for each $n\leq N$ such that

$$\sum_{i\in\mathcal{I}(f,g)}\lambda(i)\sum_{n=1}^{N}p(E_n)(u_i(f(s_n))-u_i(g(s_n)))>0.$$

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